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Tiles in quasicrystals with cubic irrationality

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Abstract
We determine completely all tiles of one-dimensional cut-and-project quasicrystals which have a cubic irrationality as self-similarity factor and are given in terms of connected rectangular acceptance windows. It is shown that there are at least five and at most nine tiles, and they are indicated explicitly depending on the lengths of the two acceptance windows in terms of which the quasicrystals are defined.

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1. Introduction

The mathematical properties of cut-and-project quasicrystals [20] or model sets [19], especially the structure of their tiling sequences, are important for applications in physics, because these point sets can be used as mathematical models for the location of atoms in quasicrystals [22]. All quasicrystals discovered experimentally so far have 5-fold [22], 8-fold [26], 10-fold [1, 3, 4] or 12-fold [7] rotational symmetry, and correspond thus to the geometries underlying the lowest order rotational symmetries related to quadratic irrational numbers.

We therefore expect that if quasicrystals related to other types of irrational numbers such as cubic irrationalities occur in nature, these are most probably also the cases corresponding to the geometries underlying the rotational symmetry of lowest order related to these irrational numbers. We therefore focus here on the case of 7-fold rotational symmetry, which is the rotational symmetry of lowest order related to a cubic irrational number. The corresponding irrationality is the Pisot number $\beta = 1 + 2\cos\left(\frac{2\pi}{7}\right)$. It is a solution of the cubic equation

$$X^3 = 2X^2 + X - 1 \qquad (1)$$

and the other two solutions of modulus smaller than 1 are

$$\beta' = 1 - 2\cos\left(\frac{3\pi}{7}\right) \doteq 0.555 \qquad \beta'' = 1 - 2\cos\left(\frac{\pi}{7}\right) \doteq -0.802. \qquad (2)$$

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In general, an n-fold symmetry corresponds to an irrational number of order $\frac{\phi(n)}{2}$ where $\phi(n)$ is Euler's totient function, that is $\phi(n)$ is the number of all positive integers k such that, $gcd\{k, n\} = 1$ and $k \leqslant n$. Thus the only cubic irrationalities arise in 7-fold and 9-fold rotational symmetries.

In the case of crystals, the coordinates of the atoms are labelled by the ring of (ordinary) integers $\mathbb{Z}$. Points of quasicrystals with 5-fold symmetry, however, are labelled by the ring of integers $\mathbb{Z}_\tau = \{a + b\tau \mid a, b \in \mathbb{Z}\}$, where $\tau = 1 + 2\cos\left(\frac{2\pi}{5}\right)$ is the golden mean satisfying the equation $x^2 = x + 1$. The ring $\mathbb{Z}_\tau$ densely covers the real line. Geometrically the ring $\mathbb{Z}_\tau$ can be obtained by the orthogonal projection of the lattice $\mathbb{Z}^2$ onto the straight line $y = \tau x$. If we do not project the whole lattice $\mathbb{Z}^2$ but only its points belonging to a given strip parallel to this line, we obtain a one-dimensional cut-and-project set Λ, which has no accumulation points. The distances between consecutive points of Λ are called tiles. A suitable choice of the strip ensures that Λ is self-similar, i.e. $\tau\Lambda \subset \Lambda$. The self-similarity plays a crucial role for the diffraction properties of Λ [5, 6].

The number of different tiles in cut-and-project sets based on a projection of the lattice $\mathbb{Z}^2$ is always 2 or 3 [12–14], a fact which is also known as the three gap theorem [23–25] in number theory.

In contrast to quasicrystals related to quadratic irrational numbers, quasicrystals with points labelled by the cubic ring of integers $\mathbb{Z}_\beta \equiv \mathbb{Z}[1, \beta, \beta^2] = \{a + b\beta + c\beta^2 \mid a, b, c \in \mathbb{Z}\}$ can be obtained only by a projection of the lattice $\mathbb{Z}^3$ onto a line. Although no generalization of the three gap theorem is known for the cut-and-project sets arising via a projection of a three-dimensional lattice, it is possible to describe the collection of tiles for any quasicrystal connected to β. The most important property of β allowing this description is that cut-and-project sets based on β are self-similar with the factor β.

First we give a formal algebraic definition of quasicrystals in terms of two acceptance windows and then we show its connection to the cut-and-project terminology.

Definition 1.1. *Let* $\mathbb{Z}_\beta := \{a + b\beta + c\beta^2 \mid a, b, c \in \mathbb{Z}\}$. *Then*

$$\Sigma(\Omega_1, \Omega_2) := \{a + b\beta + c\beta^2, |a, b, c \in \mathbb{Z}, a + b\beta' + c\beta'^2 \in \Omega_1, a + b\beta'' + c\beta''^2 \in \Omega_2\} \tag{3}$$

are called cut-and-project quasicrystals *related to the cubic irrationality β, and Ω_1 and Ω_2 are connected subsets of $\mathbb{R}$ called* acceptance windows.

The point sets $\Sigma(\Omega_1, \Omega_2)$ are cut-and-project sets [19]. Indeed, with the notation $V_1 = \langle l_1\rangle_\mathbb{R}$, $V_2 = \langle l_2, l_3\rangle_\mathbb{R}$, where

$$\begin{aligned} \vec{l}_1 &= \frac{\beta' - \beta''}{7}(\beta''\beta', -\beta'' - \beta', 1) \\ \vec{l}_2 &= \frac{\beta'' - \beta}{7}(\beta''\beta, -\beta - \beta'', 1) \\ \vec{l}_3 &= \frac{\beta - \beta'}{7}(\beta\beta', -\beta - \beta', 1) \end{aligned} \tag{4}$$

any point (a, b, c) from the orthogonal lattice $\mathbb{Z}^3$ can be expressed as

$$\begin{aligned} (a, b, c) &= (a + b\beta + c\beta^2)\vec{l}_1 + (a + b\beta' + c\beta'^2)\vec{l}_2 \\ &= (a + b\beta'' + c\beta''^2)\vec{l}_3. \end{aligned} \tag{5}$$

Thus $\mathbb{Z}_\beta$ is formed by the projections of the lattice points (a, b, c) onto V_1 according to V_2. Denoting the projection onto V_1 by π_1 and the projection onto V_2 by π_2, that is

$$V_1 \xleftarrow{\pi_1} \mathbb{Z}^3 \xrightarrow{\pi_2} V_2 \tag{6}$$

Figure 1. Illustration of the points (9).

we have a one-to-one correspondence between $\pi_1(\mathbb{Z}^3)$ and $\pi_2(\mathbb{Z}^3)$ given by

$$x \in \pi_1(\mathbb{Z}^3) \to (x', x'') \in \pi_2(\mathbb{Z}^3) \tag{7}$$

and we can express (3) in definition 1.1 equivalently as

$$\Sigma(\Omega_1, \Omega_2) = \{\pi_1(x) \mid x \in \mathbb{Z}^3, \pi_2(x) \in \Omega_1 \times \Omega_2\}. \tag{8}$$

Using results about model sets [19] we thus obtain immediately that $\Lambda := \Sigma(\Omega_1, \Omega_2)$

- is a Delone set;
- has a finite number of tiles;
- is a Meyer set, that is there exists a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$.

We are concerned here, in particular, with the *set of tiles* or minimal distances $T(\Omega_1, \Omega_2)$ in $\Sigma(\Omega_1, \Omega_2)$. An example which we will discuss explicitly is the case $\Omega_1 = \Omega_2 = [0, 1)$. The ten smallest points larger than 0 are given by

$$\beta^2, -1+\beta+3\beta^2, -1+\beta+4\beta^2, -2+\beta+5\beta^2, -2+2\beta+6\beta^2, -3+2\beta+8\beta^2, \\ -4+2\beta+10\beta^2, -4+3\beta+10\beta^2, -5+3\beta+12\beta^2, -5+3\beta+13\beta^2, \ldots \tag{9}$$

and these points are illustrated in figure 1.

We show that in this example there are seven minimal distances, that is $T([0, 1), [0, 1))$ has seven elements, and we determine them explicitly. More generally, we show that the set of tiles $T(\Omega_1, \Omega_2)$ for an arbitrary cut-and-project quasicrystal (3) has at least five and at most nine elements and we give a complete account of all possible tiles $T(\Omega_1, \Omega_2)$ depending on the acceptance windows Ω_1 and Ω_2.

The paper is organized as follows. In section 2, we show that due to translation and scaling properties of the quasicrystals (3) $T(\Omega_1, \Omega_2)$ indeed only depends on the lengths of Ω_1 and Ω_2 and that only a certain subset of lengths of acceptance windows, called the relevant area, has to be considered, because the sets $T(\Omega_1, \Omega_2)$ for arbitrary Ω_1 and Ω_2 are determined by the results for $T(\Omega_1, \Omega_2)$ with window lengths in the relevant area. In section 3, we discuss the case of $\Omega_1 = \Omega_2 = [0, 1)$ explicitly, and show how the quasicrystals (3) can be generated using a stepping function. We use these results to derive the minimal tiles in $T(\Omega_1, \Omega_2)$ for arbitrary Ω_1 and Ω_2. Furthermore, we indicate an estimate on the maximal tile and introduce the terminology of *basic quadruples*, which will be a key ingredient in our analysis of the general case. In section 4, we present our main theorem, which contains a classification of all minimal distances or tiles of quasicrystals (3) depending on the lengths of the acceptance windows in the relevant area. Based on the results in section 2, one thus obtains $T(\Omega_1, \Omega_2)$ for all quasicrystals in definition 1.1. In the conclusion, we finally compare our results with the case of quasicrystals related to quadratic irrational numbers and discuss the implications of our results for the three gap theorem in number theory.

2. Geometrically similar quasicrystals

The aim of this section is to restrict the set of quasicrystals $\Sigma(\Omega_1, \Omega_2)$ to a subset such that the collection of tiles $T(\Omega_1, \Omega_2)$ of $\Sigma(\Omega_1, \Omega_2)$ for arbitrary choices of Ω_1 and Ω_2 follows from the results for a member $\Sigma(\hat{\Omega}_1, \hat{\Omega}_2)$ of this subset via a simultaneous rescaling of all elements in the corresponding set of tiles $T(\hat{\Omega}_1, \hat{\Omega}_2)$.

In particular, we introduce an equivalence relation on the set of quasicrystals $\Sigma(\Omega_1, \Omega_2)$ as follows.

Definition 2.1. *We call two quasicrystals $\Sigma\left(\Omega_1^A, \Omega_2^A\right)$ and $\Sigma\left(\Omega_1^B, \Omega_2^B\right)$* equivalent *if they have the same set of tiles modulo rescaling by a constant factor $\lambda \in \mathbb{Z}_\beta$, that is if*

$$T\left(\Omega_1^A, \Omega_2^A\right) = \lambda T\left(\Omega_1^B, \Omega_2^B\right). \tag{10}$$

One has the following lemma.

Lemma 2.2. *Let Ω_1 and Ω_2 be bounded semiclosed intervals. $\Sigma(\Omega_1, \Omega_2)$ and $\Sigma(\Omega_1 + c, \Omega_2 + d)$ are equivalent for any $c, d \in \mathbb{R}$.*

Proof. According to [2], $\Sigma(\Omega_1, \Omega_2)$ is locally isomorphic to $\Sigma(\Omega_1 + c, \Omega_2 + d)$ for $c, d \in \mathbb{R}$. This implies that $T(\Omega_1, \Omega_2) = T(\Omega_1 + c, \Omega_2 + d)$ for $c, d \in \mathbb{R}$, and $\Sigma(\Omega_1, \Omega_2)$ and $\Sigma(\Omega_1 + c, \Omega_2 + d)$ are thus equivalent according to definition 2.1 with $\lambda = 1$, which proves the claim. □

Hence, only the lengths of Ω_1 and Ω_2 are relevant for $T(\Omega_1, \Omega_2)$.

Furthermore, one has the following proposition.

Proposition 2.3. *$\Sigma([0, l_1), [0, l_2))$ and $\Sigma([0, l_1], [0, l_2])$ for $l_1, l_2 \in \mathbb{R}$ differ in at most two points.*

Proof. Since a straight line $x =$ const or $y =$ const contains at most one point $(a', a'') \in \mathbb{Z}_\beta$, a quasicrystal with an acceptance window $[u, v) \times [s, t)$ and a quasicrystal with an acceptance window $[u, v] \times [s, t]$ differ in at most two points. □

This may change the collection of tiles by at most two new tiles which occur in $\Sigma([u, v], [s, t])$ exactly once. For example, in the quasicrystal $\Sigma([0, 1], [0, 1])$ the tile of length 1 occurs precisely once between the points 0 and 1. The same situation may occur when one considers a window of the form $(u, v) \times (s, t)$. To avoid such singularities, we consider in the following only acceptance windows of the form $[u, v) \times [s, t)$, that is we will restrict all further considerations to quasicrystals of the form $\Sigma([0, l_1), [0, l_2))$ with $l_1, l_2 \in \mathbb{R}$.

Remark. Acceptance windows $(u, v] \times (s, t]$ do not need to be considered, because they follow via $\Sigma((u, v], (s, t]) = -\Sigma([-v, -u), [-t, -s))$.

Then one has the following theorem.

Theorem 2.4. *For any $\Sigma(\Omega_1, \Omega_2)$ where Ω_1 and Ω_2 are semiclosed bounded intervals, there exists (l_1, l_2) in*

$$\mathcal{L}_{\mathrm{RA}} = (\beta', 1] \times \left(\frac{1}{\beta}, 1\right] \cup \left(\beta', \frac{\beta'}{|\beta''|}\right] \times \left(\frac{|\beta''|}{\beta}, \frac{1}{\beta}\right] \tag{11}$$

such that $\Sigma(\Omega_1, \Omega_2)$ and $\Sigma([0, l_1), [0, l_2))$ are equivalent.

Proof. Since β is the Pisot unit, that is $\beta\mathbb{Z}_\beta = \beta'\mathbb{Z}_\beta = \beta''\mathbb{Z}_\beta = \mathbb{Z}_\beta$, one has the following geometric similarities between quasicrystals with acceptance windows of different lengths:

$$\beta\Sigma(\Omega_1, \Omega_2) = \Sigma(\beta'\Omega_1, \beta''\Omega_2) \tag{12}$$

$$\beta'\Sigma(\Omega_1, \Omega_2) = \Sigma(\beta''\Omega_1, \beta\Omega_2) \tag{13}$$

$$\beta''\Sigma(\Omega_1, \Omega_2) = \Sigma(\beta\Omega_1, \beta'\Omega_2). \tag{14}$$

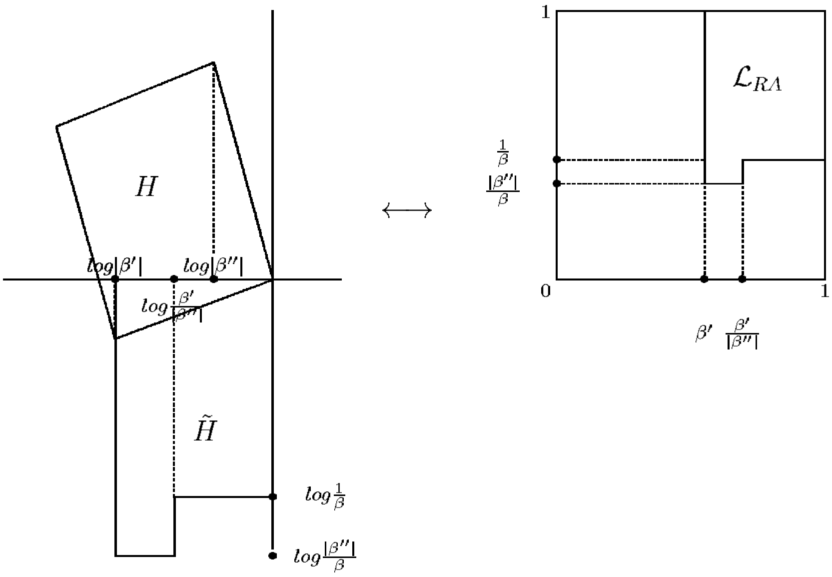

Figure 2. The sets H, $\tilde{H}$ and $\mathcal{L}_{\mathrm{RA}}$.

Due to (12)–(14), the geometrical properties of $\Sigma(\Omega_1, \Omega_2)$ coincide with the properties of quasicrystals with the acceptance windows $\beta^k\beta'^s\beta''^l\Omega_1$ and $\beta'^k\beta''^s\beta^l\Omega_2$ with $k, s, l \in \mathbb{Z}$. As $\beta\beta'\beta'' = -1$, the three equations (12)–(14) are dependent and one can without loss of generality consider a rescaling of the acceptance windows by (12) and (13) only, thus

$$(\beta')^s(\beta'')^l\Omega_1 \qquad \text{and} \qquad (\beta'')^s(\beta)^l\Omega_2. \tag{15}$$

Denoting by $l(\Omega_1)$ and $l(\Omega_2)$ the lengths of the acceptance windows, $(\log l(\Omega_1), \log l(\Omega_2))$ can be transformed into

$$(\log l(\Omega_1) + s\log\beta' + l\log|\beta''|, \log l(\Omega_2) + s\log|\beta''| + l\log\beta). \tag{16}$$

If we consider the lattice $\mathcal{L}(\vec{x}_1, \vec{x}_2) := \mathbb{Z}\vec{x}_1 + \mathbb{Z}\vec{x}_2$, with $\vec{x}_1 = (\log\beta', \log|\beta''|)$ and $\vec{x}_2 = (\log|\beta''|, \log\beta)$ then all quasicrystals are equivalent to some quasicrystals with their lengths of acceptance windows (l_1, l_2) restricted by the condition:

$$(\log l_1, \log l_2) = a_1\vec{x}_1 + a_2\vec{x}_2 \qquad 0 \leqslant a_1, a_2 < 1. \tag{17}$$

We depict the area as H on the left-hand side of figure 2. It is also possible to describe the quasicrystal with $(\log l_1, \log l_2)$ in any other area of $\mathbb{R}^2$, say $\tilde{H}$, where $\tilde{H}$ is such that there exists for each $x \in \tilde{H}$ a pair $(s, l) \in \mathbb{Z} \times \mathbb{Z}$ such that $s\vec{x}_1 + l\vec{x}_2 \in H$. For later convenience, we choose $\tilde{H}$ according to figure 2.

Furthermore, $(\log l_1, \log l_2) \in \tilde{H}$ is equivalent to $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$ as depicted on the right-hand side of figure 2. Since $\mathcal{L}_{\mathrm{RA}}$ in figure 2 corresponds to (11), this proves the claim. □

We will therefore in the following refer to $\mathcal{L}_{\mathrm{RA}}$ as the *relevant area* and restrict all our further considerations to $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$.

3. Reference quasicrystals, minimal and maximal tiles and basic quadruples

As a preparation for the main result, we need to introduce further concepts. Due to the inclusion

$$\Sigma([0, \beta'), [0, |\beta''^2 - 1|)) \subset \Sigma([0, l_1), [0, l_2)) \subset \Sigma([0, 1), [0, 1)) \tag{18}$$

for all (l_1, l_2) in the relevant area $\mathcal{L}_{\text{RA}}$ the quasicrystals $\Sigma([0, \beta'), [0, |\beta''^2 - 1|))$ and $\Sigma([0, 1), [0, 1))$ play a significant role in our considerations, because they provide estimates on the smallest and largest tiles for $\Sigma([0, l_1), [0, l_2))$ with $(l_1, l_2) \in \mathcal{L}_{\text{RA}}$. We will therefore call them *reference quasicrystals* in the following.

In particular, in this section we investigate the sets of tiles of these reference quasicrystals in detail and based on this derive results about the minimal and the maximal tiles in $T([0, l_1), [0, l_2))$ with $(l_1, l_2) \in \mathcal{L}_{\text{RA}}$. Also, we introduce the terminology of basic quadruples, which will play a key role in our proof of the main result. As we have already mentioned the collection of tiles depends only on the lengths of acceptance windows. Therefore, we will use the short-hand notation $T(l_1, l_2)$ instead of $T([0, l_1), [0, l_2))$ in the following.

3.1. *The reference quasicrystal* $\Sigma([0, 1), [0, 1))$ *and minimal tiles*

This case corresponds to the choice of lengths $(l_1, l_2) = (1, 1)$ and thus to the upper right-hand corner of $\mathcal{L}_{\text{RA}}$ in figure 2. Due to (18), it gives a lower bound on the tiles in $T(l_1, l_2)$ with $(l_1, l_2) \in \mathcal{L}_{\text{RA}}$.

Lemma 3.1. *$T(1, 1)$ consists of the following seven elements:*

$$\begin{array}{llll} \Delta_1 = \beta & \Delta_2 = \beta^2 - 1 & \Delta_3 = \beta^2 & \Delta_4 = \beta^2 + \beta \\ \Delta_5 = 2\beta^2 - 1 & \Delta_6 = 2\beta^2 + \beta - 1 & \Delta_7 = 3\beta^2 + \beta - 1. & \end{array} \tag{19}$$

Proof. Let $x_1 < x_2$ be two neighbouring points in $\Sigma([0, 1), [0, 1))$. Then $\Delta = x_2 - x_1$ is the length of the corresponding tile and Δ can be written as $\Delta = K + L\beta + M\beta^2$ with $K, L, M \in \mathbb{Z}$. Since x_1', x_2', x_1'' and $x_2'' \in [0, 1)$, both Δ' and $\Delta'' \in (-1, 1)$. At first we consider only tiles with a length less than or equal to $3\beta^2 + \beta - 1$. This leads to the following inequalities for K, L, M in $\mathbb{Z}$:

$$\begin{aligned} &0 < \Delta = K + L\beta + M\beta^2 \leqslant 3\beta^2 + \beta - 1 \\ &-1 < \Delta' = K + L\beta' + M\beta'^2 < 1 \\ &-1 < \Delta'' = K + L\beta'' + M\beta''^2 < 1. \end{aligned} \tag{20}$$

We look for solutions, that is we seek lattice points $K\vec{f}_1 + L\vec{f}_2 + M\vec{f}_3$ with $\vec{f}_1 = (1, 1, 1)$, $\vec{f}_2 = (\beta, \beta', \beta'')$ and $\vec{f}_3 = (\beta^2, \beta'^2, \beta''^2)$, inside this parallelepiped. Clearly, the set of solutions is finite. Using Maple, we obtain nine triples (K_i, L_i, M_i), $i = 1, \ldots, 9$, and thus nine corresponding candidates $B_i = K_i + L_i\beta + M_i\beta^2$ for Δ. We state them together with the approximate values for B_i, B_i' and B_i'':

$$\begin{array}{lll} B_1 = \beta \doteq 2.247 & B_1' \doteq 0.555 & B_1'' \doteq -0.802 \\ B_2 = \beta^2 - 1 \doteq 4.049 & B_2' \doteq -0.692 & B_2'' \doteq -0.357 \\ B_3 = \beta^2 \doteq 5.049 & B_3' \doteq 0.308 & B_3'' \doteq 0.643 \\ B_4 = \beta^2 + \beta \doteq 7.296 & B_4' \doteq 0.863 & B_4'' \doteq -0.159 \\ B_5 = 2\beta^2 - 1 \doteq 9.098 & B_5' \doteq -0.384 & B_5'' \doteq 0.286 \\ B_6 = 2\beta^2 + \beta - 1 \doteq 11.345 & B_6' \doteq 0.171 & B_6'' \doteq -0.516 \\ B_7 = 3\beta^2 - 1 \doteq 14.147 & B_7' \doteq -0.076 & B_7'' \doteq 0.930 \\ B_8 = 3\beta^2 + \beta - 2 \doteq 15.394 & B_8' \doteq -0.521 & B_8'' \doteq -0.872 \\ B_9 = 3\beta^2 + \beta - 1 \doteq 16.394 & B_9' \doteq 0.479 & B_9'' \doteq 0.128. \end{array} \tag{21}$$

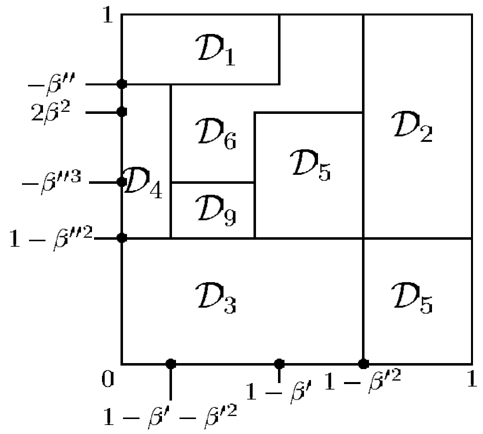

Figure 3. Partition of the acceptance window of $\Sigma([0,1),[0,1))$.

Thus, $B_1,\ldots,B_9$ are the only candidates for tiles of length smaller than or equal to $3\beta^2+\beta-1$. It remains to check which of the B_k, $k=1,\ldots,9$, are indeed tiles, that is if for a given B_k there exists a point $x\in\Sigma([0,1),[0,1))$ such that $x+B_k\in\Sigma([0,1),[0,1))$ and there is no B_j with $j<k$ such that $x+B_j\in\Sigma([0,1),[0,1))$. Since $B_1=\beta$ is the smallest candidate in (21), it is a tile if and only if there exists $a\in\mathbb{Z}_\beta$ with $(a',a'')\in[0,1)\times[0,1)$ such that $(a',a'')+(B_1',B_1'')\in[0,1)\times[0,1)$, that is such that

$$0\leqslant a'+B_1'<1 \qquad 0\leqslant a''+B_1''<1. \tag{22}$$

Since the set $\{(a',a'')\mid a\in\mathbb{Z}_\beta\}$ is dense in the plane there exist infinitely many points a in $\Sigma([0,1),[0,1))$ for which the right neighbour is $a+B_1$. Therefore B_1 is a tile, and we denote it by Δ_1. For $a\in\Sigma([0,1),[0,1))$ with right neighbour $a+\Delta_1$ we depict (a',a'') in figure 3 as the area $\mathcal{D}_1$ in the partition of the acceptance window $[0,1)\times[0,1)$.

The remaining points a of $\Sigma([0,1),[0,1))$ with $(a',a'')\notin\mathcal{D}_1$ have a right neighbour $x+B_i$, for some $i\geqslant 2$. We again start with the smallest candidate for a tile, which is B_2. B_2 corresponds to a tile in $\Sigma([0,1),[0,1))$ if there exists a pair (a',a'') in $[0,1)\times[0,1)\backslash\mathcal{D}_1$ such that $(a',a'')+(B_2',B_2'')\in[0,1)\times[0,1)$. Again we see that there exist infinitely many such pairs (depicted in figure 3 as the area $\mathcal{D}_2$) and therefore $\Delta_2:=B_2$ is also a tile of the quasicrystal.

In this way one can check that $\Delta_3:=B_3,\ldots,\Delta_6:=B_6$ are tiles in $\Sigma([0,1),[0,1))$. But the points (a',a''), to which one can add (B_7',B_7''), respectively (B_8',B_8''), such that $(a',a'')+(B_7',B_7'')$, respectively $(a',a'')+(B_8',B_8'')$, belongs to $[0,1)\times[0,1)$ have the property that there exists B_j with $j<7$ such that $(a',a'')+(B_j',B_j'')$ also belongs to $[0,1)\times[0,1)$. Therefore, B_7 and B_8 do not correspond to tiles in $\Sigma([0,1),[0,1))$.

However, $\Delta_7:=B_9$ is a tile. Since we have completely covered $[0,1)\times[0,1)$ with the areas $\mathcal{D}_1,\ldots,\mathcal{D}_7$, no other tiles longer than $3\beta^2+\beta-1$ can occur in $\Sigma([0,1),[0,1))$. □

Remark. We remark that diagrams such as figure 3 can be implemented to construct the tiling sequences of $\Sigma(\Omega_1,\Omega_2)$. We demonstrate the method here for $\Sigma([0,1),[0,1))$ based on figure 3.

Using the partition of $[0,1)\times[0,1)$ into the areas $\mathcal{D}_1,\ldots,\mathcal{D}_7$ as depicted in figure 3, define for $\Sigma([0,1),[0,1))$ the stepping function f, which assigns to a point a from the quasicrystal its right neighbour $f(a)$ as follows:

$$f:\Sigma([0,1),[0,1))\to\Sigma([0,1),[0,1)) \tag{23}$$

where

$$f(a) = a + \Delta_k \qquad \text{if} \quad (a', a'') \in \mathcal{D}_k \quad k = 1, \ldots, 7. \tag{24}$$

Starting from a point $x_0 \in \Sigma([0, 1), [0, 1))$ we can thus calculate right consecutive points of x_0 as $f(x_0), f^2(x_0), f^3(x_0), \ldots, f^n(x_0)$, or, left consecutive points of x_0 as $f^{-1}(x_0), f^{-2}(x_0), f^{-3}(x_0), \ldots, f^{-n}(x_0)$ for any n.

For example, for $x_0 = 0$ we have $(x_0', x_0'') = (0, 0) \in \mathcal{D}_3$, and the right neighbour of 0 is $x_1 = f(0) = \Delta_3 = \beta^2$. Furthermore, since $(x_1', x_1'') \doteq (0.308, 0.643) \in \mathcal{D}_6$, the next point in the quasicrystal is $x_2 = f^2(0) = x_1 + \Delta_6 = 3\beta^2 + \beta - 1$ and so on. The first 13 right neighbours of 0 in the quasicrystal $\Sigma([0, 1), [0, 1))$ are indicated in the appendix.

For quasicrystals $\Sigma([c, c + 1), [d, d + 1))$ with $c, d \in \mathbb{R}$ one obtains the corresponding tiling sequences via a modification of the stepping function by using instead of the areas $\mathcal{D}_i$ their shifted copies $(c, d) + \mathcal{D}_i$.

We use the results about the reference quasicrystal $\Sigma([0, 1), [0, 1))$ to derive the minimal tiles in $T(l_1, l_2)$ with $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$.

Let Δ be a tile between two consecutive points $x_1 < x_2$ in a quasicrystal $\Sigma([0, l_1), [0, l_2))$. Then

$$\Delta' = x_2' - x_1' \in (-l_1, l_1) \subset (-1, 1) \qquad \text{and} \qquad \Delta'' = x_2'' - x_1'' \in (-l_2, l_2) \subset (-1, 1).$$

This means that Δ is a positive element in the union of the following four quasicrystals:

$$\Delta \in \Sigma([0, 1), [0, 1)) \cup \Sigma((-1, 0], (-1, 0]) \cup \Sigma([0, 1), (-1, 0]) \cup \Sigma((-1, 0], [0, 1)).$$

These four quasicrystals thus provide candidates for tiles and for later convenience we introduce the following short-hand notation:

$$\Sigma_+^+ := \Sigma([0, 1), [0, 1)) \qquad \Sigma_-^- := \Sigma((-1, 0], (-1, 0])$$
$$\Sigma_-^+ := \Sigma([0, 1), (-1, 0]) \qquad \Sigma_+^- := \Sigma((-1, 0], [0, 1)).$$

We have already calculated the positive elements of Σ_+^+ by using the stepping function. Table A1 in the appendix contains all positive elements of length smaller than 90. Since $\Sigma_-^- = -1 + \Sigma((0, 1], (0, 1])$, we obtain the positive elements of Σ_-^- by a subtraction of -1 from Σ_+^+ and we list them in table A2 in the appendix. Elements of Σ_-^+ and Σ_+^- are listed in tables A3 and A4 in the appendix, respectively; they have been obtained via a stepping function f, where the corresponding acceptance windows have been split into appropriately shifted areas $\mathcal{D}_i$, $i = 1, \ldots, 7$.

Theorem 3.2. *Let $\Delta_{\min}(l_1, l_2)$ denote the minimal tile in $T(l_1, l_2)$ with $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$. Then one has*

$$\begin{array}{llll}
|\beta'| < l_1 \leqslant 1 & \textit{and} & |\beta''| < l_2 \leqslant 1 & \Longrightarrow \quad \Delta_{\min}(l_1, l_2) = \beta \\
|\beta'^2 - 1| < l_1 \leqslant 1 & \textit{and} & |\beta''^2 - 1| < l_2 \leqslant |\beta''| & \Longrightarrow \quad \Delta_{\min}(l_1, l_2) = \beta^2 - 1 \\
|\beta'| < l_1 \leqslant |\beta'^2 - 1| & \textit{and} & |\beta''^2| < l_2 \leqslant |\beta''| & \Longrightarrow \quad \Delta_{\min}(l_1, l_2) = \beta^2 \\
|\beta'| < l_1 \leqslant |\beta'^2 - 1| & \textit{and} & |\beta''^2 - 1| < l_2 \leqslant |\beta''^2| & \Longrightarrow \quad \Delta_{\min}(l_1, l_2) = 2\beta^2 - 1.
\end{array}$$

Proof. The smallest positive element Δ in the union of the four quasicrystals $\Sigma_+^+ \cup \Sigma_-^+ \cup \Sigma_+^- \cup \Sigma_-^-$ which is such that $|\Delta'| < l_1$ and $|\Delta''| < l_2$ is clearly the minimal distance in the quasicrystal. From the list of positive elements of the above-mentioned four quasicrystals in the appendix we thus immediately obtain the result. □

The values of the minimal elements in $T(l_1, l_2)$ depending on (l_1, l_2) in the relevant area $\mathcal{L}_{\mathrm{RA}}$ are depicted in figure 4.

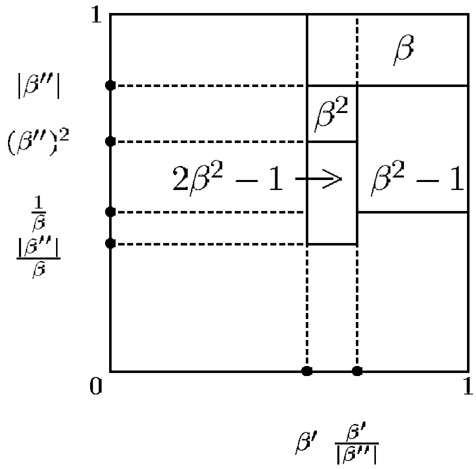

Figure 4. Partition of $\mathcal{L}_{\mathrm{RA}}$ according to the size of the minimal tiles.

3.2. The basic quadruple

A key concept in the proof of the main theorem will be basic quadruples, which we introduce in this subsection.

Definition 3.3. *Let* Δ *be a tile of a quasicrystal* $\Sigma([0, l_1), [0, l_2))$. *Then we call* $(\mathrm{sgn}(\Delta'), \mathrm{sgn}(\Delta''))$ *the* signature *of* Δ.

The smallest tile of each signature always exists in $\Sigma([0, l_1), [0, l_2))$ due to the following.

Lemma 3.4. *The first positive elements from* Σ_+^+, Σ_-^+, Σ_-^- *and* Σ_+^- *satisfying (26) belong to* $T(l_1, l_2)$.

Proof. Consider $x \in \Sigma([0, l_1), [0, l_2))$ such that $0 < x' < \varepsilon$ and $0 < x'' < \varepsilon$ for

$$\varepsilon < \min_{\Delta \in T(l_1, l_2)} \{|\Delta'|, |\Delta''|\} \tag{25}$$

and consider $x + \Delta_0$ to be the right neighbour of x. Since $|\Delta_0'|, |\Delta_0''| > \varepsilon$ and $(x + \Delta_0)' \in \Omega_1$, $(x + \Delta_0)'' \in \Omega_2$, we must have $\Delta_0' > 0$ and $\Delta_0'' > 0$. Therefore, the first positive element z of Σ_+^+ such that

$$|z'| < l_1 \qquad |z''| < l_2 \tag{26}$$

is certainly in $T(l_1, l_2)$. Since it is the element with signature $(+1, +1)$, we denote this element as $\Delta_+^+ = \Delta_+^+(l_1, l_2)$. Similarly, the first positive elements from Σ_-^+, Σ_-^- and Σ_+^- satisfying (26) belong to $T(l_1, l_2)$, which proves the claim. □

Definition 3.5. *We denote the tiles of smallest signature of each type in* $T(l_1, l_2)$ *by* Δ_+^+, Δ_-^+, Δ_-^- *and* Δ_+^-, *respectively, and we call the set* $\{\Delta_+^+, \Delta_-^+, \Delta_-^-, \Delta_+^-\}$ *the* basic quadruple.

Thus we can assign to any pair of lengths (l_1, l_2) the basic quadruple of tiles belonging to $T(l_1, l_2)$. It can easily be determined via tables A1–A4 in the appendix and we list the corresponding results in the following lemma.

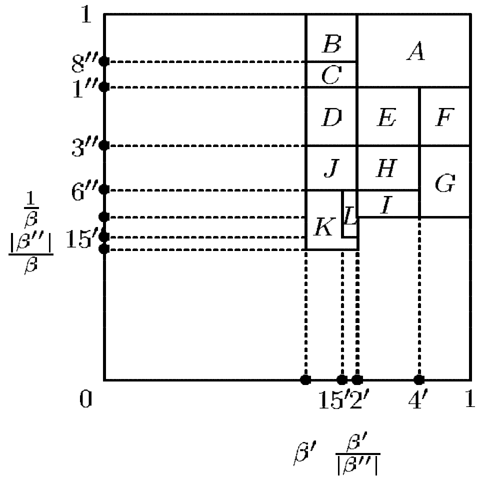

Figure 5. Partition of $\mathcal{L}_{\mathrm{RA}}$ according to the basic quadruples.

Lemma 3.6. *In the areas A–L in the partition of $\mathcal{L}_{\mathrm{RA}}$ shown in figure 5 the set of basic quadruples coincides and is given by the following list:*

Area:	++	+−	−+	−−
A	3	1	5	2
B	3	1	5	8
C	3	1	5	11
D	3	6	5	11
E	3	6	5	2
F	3	4	5	2
G	9	4	5	2
H	9	6	5	2
I	9	15	5	2
J	9	6	5	11
K	9	20	5	11
L	9	15	5	11

(27)

where the numbers in the table refer to the numbers of the corresponding tiles in the list (31) in the appendix.

3.3. The reference quasicrystal $\Sigma([0, \beta'), [0, |\beta''^2 - 1|))$ and an estimate on the maximal tile

The quasicrystal corresponding to the lower left-hand corner of the relevant area $\mathcal{L}_{\mathrm{RA}}$ is $\Sigma([0, \beta'), [0, |\beta''^2 - 1|))$, and due to (18) it provides estimates on the maximal tile in $T([0, l_1), [0, l_2))$ with $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$. We therefore discuss this case separately here.

Lemma 3.7. *The set $T(\beta', |\beta''^2 - 1|)$ consists of the basic quadruple*

$$\Delta_+^+ = 3\beta^2 + \beta - 1 \doteq 16.394 \qquad (\Delta_+^+)' \doteq 0.479 \qquad (\Delta_+^+)'' \doteq 0.128$$
$$\Delta_-^+ = 7\beta^2 + 2\beta - 3 \doteq 36.84 \qquad (\Delta_-^+)' \doteq 0.27 \qquad (\Delta_-^+)'' \doteq -0.10$$
$$\Delta_+^- = 2\beta^2 - 1 \doteq 9.10 \qquad (\Delta_+^-)' \doteq -0.38 \qquad (\Delta_+^-)'' \doteq 0.29$$
$$\Delta_-^- = 4\beta^2 + \beta - 2 \doteq 20.44 \qquad (\Delta_-^-)' \doteq -0.21 \qquad (\Delta_-^-)'' \doteq -0.23$$

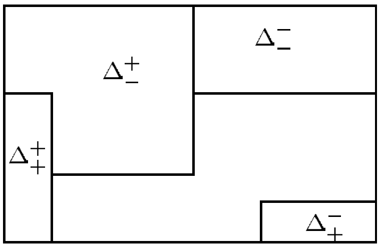

Figure 6. The basic quadruple for $\Sigma([0, \beta'), [0, |\beta''^2 - 1|))$.

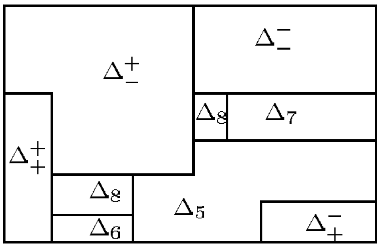

Figure 7. Partition of the acceptance window for the reference quasicrystal $\Sigma([0, \beta'), [0, |\beta''^2 - 1|))$.

and furthermore the following tiles:

$$\begin{array}{lll}
\Delta_5 = 9\beta^2 + 2\beta - 4 \doteq 45.93 & (\Delta_5)' \doteq 0.479 & (\Delta_5)'' \doteq 0.128 \\
\Delta_6 = 12\beta^2 + 3\beta - 5 \doteq 36.84 & (\Delta_6)' \doteq 0.36 & (\Delta_6)'' \doteq 0.31 \\
\Delta_7 = 13\beta^2 + 3\beta - 6 \doteq 66.38 & (\Delta_7)' \doteq -0.33 & (\Delta_7)'' \doteq -0.05 \\
\Delta_8 = 16\beta^2 + 4\beta - 7 \doteq 82.76 & (\Delta_8)' \doteq 0.15 & (\Delta_8)'' \doteq 0.08.
\end{array}$$

Proof. We determine the tiles as before from the lists of elements in Σ_+^+, Σ_-^+, Σ_+^- and Σ_-^- according to the graphical method used in the case of $\Sigma([0, 1), [0, 1))$. The basic quadruple is indicated in figure 6. For the quasicrystal points a with (a', a'') in the areas Δ_+^+, Δ_+^-, Δ_-^+ and Δ_-^-, we thus already know the neighbouring points.

For the remaining points $a \in \Sigma([0, \beta'), [0, |\beta''^2 - 1|))$ we deduce the corresponding tiles by reading off the smallest possible tile, say Δ_5, from the lists in the appendix, such that $a + \Delta_5 \in \Sigma([0, \beta'), [0, |\beta''^2 - 1|))$. We obtain $\Delta_5 = 9\beta^2 + 2\beta - 4$ and augment the set of points a for which the neighbours are already determined by the corresponding area in window space, that is the area corresponding to pairs (a', a'') to which one may add the vector (Δ_5', Δ_5''). We repeat this procedure until we cover the whole acceptance window. We thus obtain figure 7.

It shows the area covered by the eight tiles of the quasicrystal $\Sigma([0, \beta'), [0, |\beta''^2 - 1|))$, and thus proves the claim. $\square$

Due to the inclusion (18) lemma 3.7 provides an estimate on the maximal tile $\Delta_{\max}(l_1, l_2)$ in each quasicrystal $\Sigma([0, l_1), [0, l_2))$ with $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$.

Corollary 3.8. *The maximal tile in* $T(l_1, l_2)$ *with* $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$ *is bounded by*

$$\Delta_{\max}(l_1, l_2) \leqslant \Delta_{\max}(\beta', |\beta''^2 - 1|) = 16\beta^2 + 4\beta - 7 \doteq 82.76. \tag{28}$$

4. The set of tiles in the general case

The estimate on the maximal tile at the end of the previous section limits the set of possible tiles $T(l_1, l_2)$ to those tiles in the lists in the appendix which are smaller than $16\beta^2 + 4\beta - 7$. This condition is not sufficient, and we can indeed exclude further candidates for tiles from these lists as follows.

Lemma 4.1. *Let* $\Delta_1, \Delta_2 \in \Sigma_+^+$ *such that*

$$0 < \Delta_1 < \Delta_2 \qquad 0 < \Delta_1' < \Delta_2' \qquad 0 < \Delta_1'' < \Delta_2''.$$

Then $\Delta_2 \notin T(l_1, l_2)$.

Proof. If $x, x + \Delta_2 \in \Sigma([0, l_1), [0, l_2))$, then also $x + \Delta_1 \in \Sigma([0, l_1), [0, l_2))$ and thus $x + \Delta_2$ is not an immediate neighbouring point of x. Thus $\Delta_2 \notin T(l_1, l_2)$. □

For example, such a situation occurs for $\Delta_1, \Delta_2 \in \Sigma_+^+$ with

$$16.394 \doteq \Delta_1 = 3\beta^2 + \beta - 1 < 4\beta^2 + \beta - 1 = \Delta_2 \doteq 21.44$$

because

$$\Delta_1' \doteq 0.479 < \Delta_2' \doteq 0.79 \qquad \text{and} \qquad \Delta_1'' \doteq 0.128 < \Delta_2'' \doteq 0.77.$$

Analogous rules exist for Σ_-^+, Σ_+^- and Σ_-^-. For example, from the list corresponding to Σ_-^+ we can exclude Δ_2 if there exists Δ_1 such that

$$0 < \Delta_1 < \Delta_2 \qquad 0 < \Delta_1' < \Delta_2' \qquad \text{and} \qquad 0 > \Delta_1'' > \Delta_2''. \tag{29}$$

Applying these rules to the elements from Σ_+^+, Σ_+^-, Σ_-^+ and Σ_-^-, we reduce the list of possible tiles to the following candidates.

Lemma 4.2. $T(l_1, l_2)$ *with* $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$ *is a subset of the tiles in the following list:*

$$\begin{array}{ll}
\textit{Six in } \Sigma_+^+: & \beta^2, 3\beta^2 + \beta - 1, 5\beta^2 + \beta - 2, 10\beta^2 + 3\beta - 4, 12\beta^2 + 3\beta - 5, \\
& 16\beta^2 + 4\beta - 7 \\
\textit{Six in } \Sigma_+^-: & 2\beta^2 - 1, 3\beta^2 - 1, 6\beta^2 + \beta - 3, 7\beta^2 + \beta - 3, 9\beta^2 + 2\beta - 4, \\
& 14\beta^2 + 3\beta - 6 \\
\textit{Six in } \Sigma_-^+: & \beta, \beta^2 + \beta, 2\beta^2 + \beta - 1, 5\beta^2 + 2\beta - 2, 7\beta^2 + 2\beta - 3, \\
& 11\beta^2 + 3\beta - 5 \\
\textit{Seven in } \Sigma_-^-: & \beta^2 - 1, 3\beta^2 + \beta - 2, 4\beta^2 + \beta - 2, 6\beta^2 + 2\beta - 3, 10\beta^2 + 2\beta - 5, \\
& 13\beta^2 + 3\beta - 6, 15\beta^2 + 4\beta - 7.
\end{array} \tag{30}$$

They correspond to the following objects in table A5 in the appendix:

$$\begin{array}{ll}
++: & 3, 9, 14, 31, 35, 48 \\
--: & 2, 8, 11, 17, 28, 38, 44 \\
+-: & 1, 4, 6, 15, 20, 33 \\
-+: & 5, 7, 16, 19, 26, 41.
\end{array} \tag{31}$$

For the sake of keeping the notation brief we will use in the following the numbers in table A5 in the appendix instead of the tiles themselves.

The strategy of the proof of the main result will consist in finding a suitable partition of the relevant area $\mathcal{L}_{\mathrm{RA}}$ such that the set of tiles coincides in each subset of this partition. For this we need to set up further definitions and results.

Definition 4.3. *We introduce the following* ordering *on* $\mathbb{R} \times \mathbb{R}$*:*

$$(a, b) > (c, d) \qquad \textit{if} \quad a > c \quad \textit{and} \quad b > d. \tag{32}$$

Proposition 4.4. *Let* Δ *be a tile in the list (31). Then we have: if* $\Delta \in T(l_1, l_2)$*, then* $\Delta \in T(\hat{l}_1, \hat{l}_2)$ *for all* $(\hat{l}_1, \hat{l}_2)$ *with* $(|\Delta'|, |\Delta''|) < (\hat{l}_1, \hat{l}_2) < (l_1, l_2)$.

Proof. If $\Delta \in \Sigma(\hat{l}_1, \hat{l}_2)$ then in the geometric method explained and implemented earlier it reserves space in the acceptance window $[0, \hat{l}_1) \times [0, \hat{l}_2)$. Since the process of shrinking the acceptance window is continuous, this space cannot be taken by a smaller tile if the window lengths are decreased to some $(\hat{l}_1, \hat{l}_2)$. Thus the tile remains present for all $(\hat{l}_1, \hat{l}_2) > (|\Delta'|, |\Delta''|)$ until it ceases to exist at $(|\Delta'|, |\Delta''|)$. □

Corollary 4.5. *If any of 5, 9, 11, 20, 26, 33 or 38 in table A5 in the appendix is a tile for* $\Sigma([0, l_1), [0, l_2))$ *with* $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$*, then it is also a tile for all* $\Sigma([0, \hat{l}_1), [0, \hat{l}_2))$ *with* $(\hat{l}_1, \hat{l}_2) \in \mathcal{L}_{\mathrm{RA}}$ *and* $(\hat{l}_1, \hat{l}_2) < (l_1, l_2)$.

Furthermore, we have the following proposition.

Proposition 4.6. *Let* $\Sigma([0, l_1), [0, l_2))$ *with* $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$ *and let* $\Delta_2, \Delta_3 \in T(l_1, l_2)$ *be such that the signatures of* Δ_2 *and* Δ_3 *differ only in the first (second) place, and let* $\Delta_1 := \Delta_2 + \Delta_3$*. Then a necessary condition for* $\Delta_1 \in T(l_1, l_2)$ *is*

$$|\Delta_2'| + |\Delta_3'| > l_1 \qquad (|\Delta_2''| + |\Delta_3''| > l_2). \tag{33}$$

Proof. Suppose that $|\Delta_2'| + |\Delta_3'| < l_1$. Then (Δ_2', Δ_2'') and (Δ_3', Δ_3'') cover the area in window space where (Δ_1', Δ_1'') would be located, and thus $\Delta_1 \notin T(l_1, l_2)$. □

Note that the condition in proposition 4.6 is only necessary; it is sufficient only if there is no smaller tile covering the corresponding area in window space.

As an example consider tile 7 in the list (31), which is the sum of tile 5 and tile 3. From proposition 4.6 we obtain as a necessary condition for the occurrence of tile 7 in the quasicrystal that $l_1 < 2'$ and thus tile 7 can only appear when tile 2 disappears. Since there is no smaller tile occupying the corresponding area in window space, tile 7 indeed appears precisely when tile 2 disappears when shrinking the size of the acceptance windows. Thus the subset of $\mathcal{L}_{\mathrm{RA}}$ corresponding to the lengths of acceptance windows of quasicrystals in which tile 7 appears is $(7'', 1) \times (\beta', 2')$, where the value of β' in the second interval stems from the corresponding boundary of the relevant area $\mathcal{L}_{\mathrm{RA}}$.

Similarly, from the fact that tile 6 is a sum of tiles 2 and 4 and tile 8 is the sum of tiles 2 and 6 one can conclude that a necessary condition for tile 8 to exist is that tile 4 is not present.

For the convenience of the reader, we recall that $\mathcal{L}_{\mathrm{RA}}$ denotes the relevant area introduced in (11) in theorem 2.4, and that $T(l_1, l_2)$ is the short-hand notation for the set of tiles of a quasicrystal with acceptance windows of lengths l_1 and l_2, respectively, as introduced at the beginning of section 3.

Theorem 4.7. *Let the relevant area* $\mathcal{L}_{\mathrm{RA}}$ *be divided into 21 areas as depicted in figure 8. If the pair* (l_1, l_2)*, that characterizes the quasicrystal with window lengths* l_1 *and* l_2*, respectively, belongs to the interior of any such area, then the set of tiles* $T(l_1, l_2)$ *occurring in this quasicrystal is given according to the following list:*

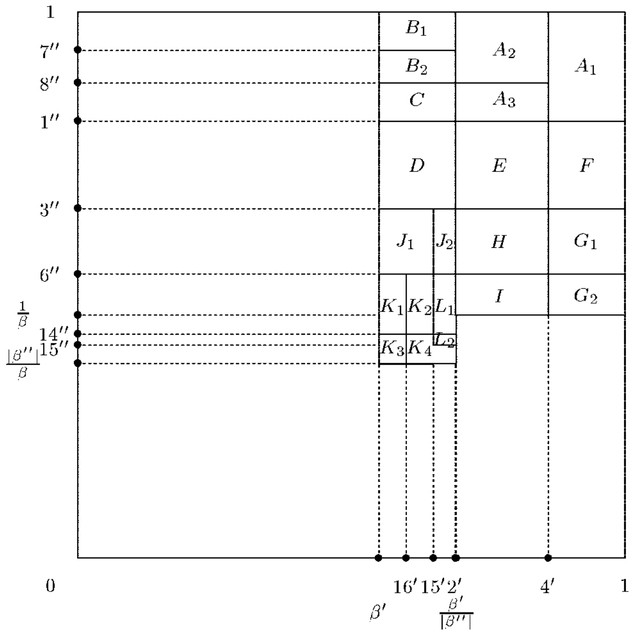

Figure 8. Partition of $\mathcal{L}_{\mathrm{RA}}$ according to theorem 4.1.

Area	*Set of tiles*	*No of tiles*
A_1	1, 2, 3, 4, 5, 6, 9	*Seven tiles*
A_2	1, 2, 3, 5, 6, 8, 9, 11	*Eight tiles*
A_3	1, 2, 3, 5, 6, 9, 11, 14	*Eight tiles*
B_1	1, 3, 5, 6, 7, 8, 9, 11, 14	*Nine tiles*
B_2	1, 3, 5, 6, 8, 9, 11, 14, 20	*Nine tiles*
C	1, 3, 5, 6, 9, 11, 14, 20	*Eight tiles*
D	3, 5, 6, 9, 11, 14, 20	*Seven tiles*
E	2, 3, 5, 6, 9, 11, 14	*Seven tiles*
F	2, 3, 4, 5, 6, 9, 11	*Seven tiles*
G_1	2, 4, 5, 6, 9, 11, 15	*Seven tiles*
G_2	2, 4, 5, 9, 11, 15, 16, 20	*Eight tiles*
H	2, 5, 6, 9, 11, 14, 15, 20	*Eight tiles*
I	2, 5, 9, 11, 14, 15, 16, 20, 26	*Nine tiles*
J_1	5, 6, 9, 11, 14, 20, 26	*Seven tiles*
J_2	5, 6, 9, 11, 14, 15, 20	*Seven tiles*
K_1	5, 9, 11, 14, 20, 26, 35	*Seven tiles*
K_2	5, 9, 11, 14, 16, 20, 26	*Seven tiles*
K_3	5, 9, 11, 20, 26, 35, 38, 48	*Eight tiles*
K_4	5, 9, 11, 16, 20, 26, 38	*Seven tiles*
L_1	5, 9, 11, 14, 15, 16, 20, 26	*Eight tiles*
L_2	5, 9, 11, 15, 16, 20, 26, 38	*Eight tiles*

(34)

where the numbers in the second column refer to the numbers of the tiles in table A5 in the appendix. If the pair (l_1, l_2) *belongs to the boundaries of several areas, then the set of tiles* $T(l_1, l_2)$ *is given as the intersection of the sets of tiles corresponding to the adjacent areas.*

Note that the areas in figure 8 are labelled as A_1–L_2, where the letters refer to the partition according to the basic quadruples displayed in figure 7.

Corollary 4.8. *$T(l_1, l_2)$ with $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$ is a subset of the tiles in the following list (numbers referring again to the list (31)):*

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 15, 16, 20, 26, 35, 38, 48. \tag{35}$$

Proof. We start by introducing a partition $\mathcal{P}$ of $\mathcal{L}_{\mathrm{RA}}$, which refines the partition of $\mathcal{L}_{\mathrm{RA}}$ related to the basic quadruples in figure 5 . $\mathcal{P}$ is constructed by taking the intersection of the area $\mathcal{L}_{\mathrm{RA}}$ (located in the (x, y)-plane) with the lines corresponding to the vanishing of tiles in the set of possible candidates in lemma 4.2. These are given in ascending order by the following lines:

- Horizontal lines

$$\begin{aligned} y &= |15''| = |5\beta''^2 + 2\beta'' - 2| \\ y &= |14''| = |5\beta''^2 + \beta'' - 2| \\ y &= |16''| = |6\beta''^2 + \beta'' - 3| \\ y &= |44''| = |15\beta''^2 + 4\beta'' - 7| \\ y &= |41''| = |14\beta''^2 + 3\beta'' - 6| \\ y &= |13''| = |5\beta''^2 + \beta'' - 3| \end{aligned} \tag{36}$$

$$\begin{aligned} y &= |19''| = |7\beta''^2 + \beta'' - 3| \\ y &= |17''| = |6\beta''^2 + 2\beta'' - 3| \\ y &= |1''| = |\beta''| \\ y &= |8''| = |3\beta''^2 + \beta'' - 2| \\ y &= |7''| = |3\beta''^2 - 1|. \end{aligned} \tag{37}$$

- Vertical lines

$$\begin{aligned} x &= |16'| = |6\beta'^2 + \beta'' - 3| \\ x &= |15'| = |5\beta'^2 + 2\beta'' - 2| \\ x &= |2'| = |\beta'^2 - 1| \\ x &= |31'| = |10\beta'^2 + 3\beta'' - 4| \\ x &= |28'| = |10\beta'^2 + 2\beta'' - 5| \\ x &= |4'| = |\beta'^2 + \beta''|. \end{aligned} \tag{38}$$

Then the strategy of the proof is as follows. In the interior of each area Λ_i in the partition $\mathcal{P}$ we choose a point $\left(l_1^i, l_2^i\right)$, called the *representative point* of Λ_i, which corresponds to the quasicrystal $\Sigma\left(\left[0, l_1^i\right), \left[0, l_2^i\right)\right)$. For this quasicrystal, we use the graphical method explained and implemented before to obtain the corresponding set of tiles $T\left(l_1^i, l_2^i\right)$. We then introduce a more coarse-grained partition $\mathcal{P}'$ by forming the union of those areas for which the sets of tiles $T\left(l_1^i, l_2^i\right)$ of their representative points coincide. The partition $\mathcal{P}'$ obtained in this way is illustrated in figure 8.

It remains to show that the set of tiles is invariant for each area in the partition $\mathcal{P}'$, that is that the boundaries of the areas in this partition indeed mark the borderline at which the sets of tiles change. But due to proposition 4.4 we know that if Δ is a tile and $\Delta \notin T(\hat{l}_1, \hat{l}_2)$, then $\Delta \notin T(l_1, l_2)$ for $(l_1, l_2) > (\hat{l}_1, \hat{l}_2)$. Furthermore, we know that if $\Delta \in T(\hat{l}_1, \hat{l}_2)$,

Table 1. Change of tiles upon transition between adjacent areas in $\mathcal{P}'$.

Transition	Tile lost	Tile(s) gained	Transition	Tile lost	Tile(s) gained
$A_1\|A_2$	4	8, 11	$G_1\|H$	4	14, 20
$A_1\|A_3$	4	11, 14	$H\|J_2$	2	∅
$A_2\|A_3$	8	14	$J_2\|J_1$	15	26
$A_2\|B_1$	2	7, 14	$G_1\|G_2$	6	16, 20
$A_2\|B_2$	2	14, 20	$H\|I$	6	16, 26
$B_1\|B_2$	7	20	$J_2\|L_1$	6	16, 26
$B_2\|C$	8	∅	$J_1\|K_2$	6	16
$A_3\|C$	2	20	$J_1\|K_1$	6	35
$C\|D$	1	∅	$G_2\|I$	4	14, 26
$A_3\|E$	1	∅	$I\|L_1$	2	∅
$A_1\|F$	1	11	$L_1\|K_2$	15	∅
$F\|E$	4	14	$K_2\|K_1$	16	35
$E\|D$	2	20	$K_1\|K_3$	14	38, 48
$F\|G_1$	3	15	$K_2\|K_4$	14	38
$E\|H$	3	15, 20	$L_1\|L_2$	14	38
$D\|J_1$	3	26	$L_2\|K_4$	15	∅
$D\|J_2$	3	15	$K_4\|K_3$	16	35, 48

then $\Delta \in T(l_1, l_2)$ for all $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$ with $(|\Delta'|, |\Delta''|) < (l_1, l_2) < (\hat{l}_1, \hat{l}_2)$. Thus any $T(l_1, l_2)$ for $(l_1, l_2) \in \mathcal{L}_{\mathrm{RA}}$ corresponds to a set $T\left(l_1^i, l_2^i\right)$, where $\left(l_1^i, l_2^i\right)$ is one of the reference points.

It thus only remains to show that the change of tiles happens precisely at the boundaries of the areas in $\mathcal{P}'$. To prove this, we do a case study based on proposition 4.6. As a preparation, we collect in table 1 the information about the change of tiles between quasicrystals corresponding to reference points in adjacent areas, say A, B, in the partition $\mathcal{P}'$. In particular, the notation $A|B$ denotes a transition from $T(l_1, l_2)$ with (l_1, l_2) in the area A to (l_1, l_2) in the area B.

The fact that transitions take place in all these cases precisely at the boundary between the corresponding areas in the partition $\mathcal{P}'$ then follows via proposition 4.6 from table 2. Table 2 shows that, when shrinking the size of the acceptance windows, the loss of the tiles (listed in the middle column in table 1) corresponds to the gain of the tiles listed in the right column of table 1.

Together with the fact that, by definition of the partition $\mathcal{P}'$, the loss of tiles coincides with the boundaries of the areas in $\mathcal{P}'$, this shows that the transitions take place precisely at the borderline between the areas in the partition $\mathcal{P}'$, which thus proves the theorem. □

It is apparent from the proof that the boundaries and corner points of the areas in the partition $\mathcal{P}'$ are singular cases. We consider two characteristic examples.

- $T(l_1, l_2)$ corresponding to the edges of areas in the partition $\mathcal{P}'$. According to theorem 4.7, the corresponding sets of tiles are obtained as the intersection of the sets of tiles corresponding to the adjacent areas which have this edge in common. For example, $\Sigma([0, l_1), [0, l_2))$ with $(l_1, l_2) \in A_1 \cap A_2$ is a six-tile quasicrystal with $T(l_1, l_2) = \{1, 2, 3, 5, 6, 9\}$ (numbers again referring to the list (31)).
- $T(l_1, l_2)$ corresponding to the corner points of an area in the partition $\mathcal{P}'$. Again according to theorem 4.7, the corresponding sets of tiles are obtained as an intersection of the sets corresponding to the adjacent areas which have this corner point in common. For example, $\Sigma([0, l_1), [0, l_2))$ with $(l_1, l_2) \in \cap_i K_i$ (see figure 8) is a five-tile quasicrystal with $T(l_1, l_2) = \{5, 9, 11, 20, 26\}$.

Table 2. Identities sustaining the fact that tiles are lost at the border line between adjacent areas in $\mathcal{P}'$.

Transition	Follows from
$A_1\|A_2$	$\|2'\| + \|6'\| = \|4'\|$
$A_1\|A_3$	$\|5'\| + \|9'\| = \|4'\|$
$A_2\|A_3$	$\|3''\| + \|11''\| = \|8''\|$
$A_2\|B_1$	$\|3'\| + \|5'\| = \|2'\|$
$A_2\|B_2$	$\|3'\| + \|5'\| = \|9'\| + \|11'\| = \|2'\|$
$B_1\|B_2$	$\|6''\| + \|14''\| = \|7''\|$
$B_2\|C$	Trivial
$A_3\|C$	$\|9'\| + \|11'\| = \|2'\|$
$C\|D$	Trivial
$A_3\|E$	Trivial
$A_1\|F$	$\|5''\| + \|6''\| = \|1''\|$
$F\|E$	$\|5'\| + \|9'\| = \|4'\|$
$E\|D$	$\|9'\| + \|11'\| = \|2'\|$
$F\|G_1$	$\|9''\| + \|6''\| = \|3''\|$
$E\|H$	$\|11''\| + \|14''\| = \|9''\| + \|6''\| = \|3''\|$
$D\|J_1$	$\|11''\| + \|14''\| = \|3''\|$
$D\|J_2$	$\|9''\| + \|6''\| = \|3''\|$
$G_1\|H$	$\|9'\| + \|5'\| = \|15'\| + \|11'\| = \|4'\|$
$H\|J_2$	Trivial
$J_2\|J_1$	$\|20'\| + \|5'\| = \|15'\|$
$G_1\|G_2$	$\|11''\| + \|5''\| = \|15''\| + \|9''\| = \|6''\|$
$H\|I$	$\|11''\| + \|5''\| = \|6''\|$
$J_2\|L_1$	$\|11''\| + \|5''\| = \|6''\|$
$J_1\|K_2$	$\|11''\| + \|5''\| = \|6''\|$
$J_1\|K_1$	$\|14''\| + \|20''\| = \|6''\|$
$G_2\|I$	$\|5'\| + \|9'\| = \|4'\|$
$I\|L_1$	Trivial
$L_1\|K_2$	Trivial
$K_2\|K_1$	$\|9'\| + \|26'\| = \|16'\|$
$K_1\|K_3$	$\|26''\| + \|11''\| = \|14''\|$
$K_2\|K_4$	$\|26''\| + \|11''\| = \|14''\|$
$L_1\|L_2$	$\|26''\| + \|11''\| = \|14''\|$
$L_2\|K_4$	Trivial
$K_4\|K_3$	$\|26'\| + \|9'\| = \|20'\| + \|38'\| = \|16'\|$

A case study shows that indeed five is the smallest number of tiles that can occur in any quasicrystal (1.1).

Finally we have the following corollary.

Corollary 4.9. *For any $T(l_1, l_2)$ with $l_1 > 0$ and $l_2 > 0$ there exist s, t $\in \mathbb{Z}$ and $T(\hat{l}_1, \hat{l}_2)$ with $(\hat{l}_1, \hat{l}_2) \in \mathcal{L}_{\mathrm{RA}}$ such that*

$$T(l_1, l_2) = \beta^s(\beta')^l T(\hat{l}_1, \hat{l}_2). \tag{39}$$

Proof. It follows via (12), (13) and (15) from theorem 4.7. □

5. Conclusion

We have provided the first analysis of the set of tiles of a cut-and-project quasicrystal or model set related to a cubic irrational number. Our analysis has shown that in the case of the irrational number β, which is related to 7-fold rotational symmetry, there are always at least five and at most nine different tiles, where the sets of tiles with cardinality 5 describe singular cases in the sense that they occur only for a finite number of points (l_1, l_2) in the relevant area $\mathcal{L}_{\mathrm{RA}}$, while all other cases occur infinitely many times in $\mathcal{L}_{\mathrm{RA}}$.

Furthermore, one observes that for any choice of acceptance window all tiles are integer linear combination of a basic quadruple of tiles and that due to the structure of these basic quadruples there are either three or four building blocks in each set of tiles $T(l_1, l_2)$ from which the whole set is constructed. This result is in contrast to the situation characteristic of quadratic irrational numbers where either two or three different tiles exist, and where we always have two building blocks.

The gaps in tiling sequences of quasicrystals related to quadratic irrational numbers correspond to the so-called three gap theorem in number theory [23–25], which has implications not only for the study of quasicrystals but also for various areas in mathematical physics, chemistry and computer science. For example, it has applications in the theory of dynamical systems, where it can be used to determine recurrence times [11, 15], in the description of molecular vibrations [24, 25], in multiplicative hashing [8, 10], in mathematical models of the ventricular parasystole [9] or in a study of phyllotaxis [21]. Our results in this paper correspond to a generalization of the three gap theorem to two dimensions for a particular choice of irrational numbers and we thus expect that it has also applications beyond the theory of quasicrystals, for example, in the areas listed above. We finally point out that until now, there have only existed numerical results [24, 25] concerning generalizations of the three gap theorem to higher dimensions and that this is the first analytical result.

Appendix

In this section we use the notation Δ_i, $i = 1, \ldots, 7$, for the tiles of $\Sigma([0, 1), [0, 1))$ according to lemma 3.1.

Table A1. The points of $\Sigma([0, 1), [0, 1)) = \Sigma_+^+$.

Quasicrystal point x	Approximate value of x	Approximate value of (x', x'')	Distance to the right neighbour
0	0	(0, 0)	Δ_3
β^2	5.049	(0.308, 0.643)	Δ_6
$3\beta^2 + \beta - 1$	16.394	(0.479, 0.128)	Δ_3
$4\beta^2 + \beta - 1$	21.44	(0.79, 0.77)	Δ_2
$5\beta^2 + \beta - 2$	25.49	(0.09, 0.41)	Δ_4
$6\beta^2 + 2\beta - 2$	32.79	(0.96, 0.25)	Δ_5
$8\beta^2 + 2\beta - 3$	41.89	(0.57, 0.54)	Δ_5
$10\beta^2 + 2\beta - 4$	50.98	(0.19, 0.83)	Δ_1
$10\beta^2 + 3\beta - 4$	53.23	(0.74, 0.03)	Δ_5
$12\beta^2 + 3\beta - 5$	62.33	(0.36, 0.31)	Δ_3
$13\beta^2 + 3\beta - 5$	67.38	(0.67, 0.95)	Δ_6
$15\beta^2 + 4\beta - 6$	78.72	(0.84, 0.43)	Δ_2
$16\beta^2 + 4\beta - 7$	82.76	(0.15, 0.08)	Δ_3
$17\beta^2 + 4\beta - 7$	87.81	(0.46, 0.77)	Δ_6

Table A2. The points of $\Sigma((-1,0],(-1,0]) = \Sigma_{-}^{-}$.

Quasicrystal point x	Approximate value of x	Approximate value of (x', x'')	Distance to the right neighbour
0	0	(0, 0)	Δ_2
$\beta^2 - 1$	4.049	(−0.69, −0.36)	Δ_6
$3\beta^2 + \beta - 2$	15.394	(−0.521, −0.872)	Δ_3
$4\beta^2 + \beta - 2$	20.44	(−0.21, −0.23)	Δ_2
$5\beta^2 + \beta - 3$	24.49	(−0.91, −0.59)	Δ_4
$6\beta^2 + 2\beta - 3$	31.79	(−0.04, −0.75)	Δ_5
$8\beta^2 + 2\beta - 4$	40.89	(−0.43, −0.46)	Δ_5
$10\beta^2 + 2\beta - 5$	49.98	(−0.81, −0.17)	Δ_1
$10\beta^2 + 3\beta - 5$	52.23	(−0.26, −0.97)	Δ_5
$12\beta^2 + 3\beta - 6$	61.33	(−0.64, −0.69)	Δ_3
$13\beta^2 + 3\beta - 6$	66.38	(−0.33, −0.05)	Δ_6
$15\beta^2 + 4\beta - 7$	77.72	(−0.16, −0.57)	Δ_2
$16\beta^2 + 4\beta - 8$	81.76	(−0.85, −0.92)	Δ_3
$17\beta^2 + 4\beta - 8$	86.81	(−0.54, −0.28)	Δ_6

Table A3. The points of $\Sigma((-1,0],[0,1)) = \Sigma_{+}^{-}$.

Quasicrystal point x	Approximate value of x	Approximate value of (x', x'')	Distance to the right neighbour
0	0	(0, 0)	Δ_5
$2\beta^2 - 1$	9.098	(−0.384, 0.286)	Δ_3
$3\beta^2 - 1$	14.147	(−0.076, 0.930)	Δ_2
$4\beta^2 - 2$	18.20	(−0.76, 0.57)	Δ_6
$6\beta^2 + \beta - 3$	29.54	(−0.60, 0.06)	Δ_3
$7\beta^2 + \beta - 3$	34.59	(−0.29, 0.70)	Δ_2
$8\beta^2 + \beta - 4$	38.64	(−0.98, 0.34)	Δ_3
$9\beta^2 + \beta - 4$	43.69	(−0.67, 0.986)	Δ_1
$9\beta^2 + 2\beta - 4$	45.93	(−0.12, 0.18)	Δ_5
$11\beta^2 + 2\beta - 5$	55.03	(−0.50, 0.47)	Δ_5
$13\beta^2 + 2\beta - 6$	64.13	(−0.88, 0.76)	Δ_4
$14\beta^2 + 3\beta - 6$	71.43	(−0.02, 0.60)	Δ_2
$15\beta^2 + 3\beta - 7$	75.47	(−0.71, 0.24)	Δ_3
$16\beta^2 + 3\beta - 7$	80.52	(−0.40, 0.88)	Δ_6

Table A4. The points of $\Sigma([0,1),(-1,0]) = \Sigma_{-}^{+}$.

Quasicrystal point x	Approximate value of x	Approximate value of (x', x'')	Distance to the right neighbour
0	0	(0, 0)	Δ_1
β	2.247	(0.555, −0.802)	Δ_3
$\beta^2 + \beta$	7.30	(0.86, −0.16)	Δ_2
$2\beta^2 + \beta - 1$	11.345	(0.171, −0.516)	Δ_7
$5\beta^2 + 2\beta - 2$	27.74	(0.65, −0.39)	Δ_5
$7\beta^2 + 2\beta - 3$	36.84	(0.27, −0.10)	Δ_1
$7\beta^2 + 3\beta - 3$	39.08	(0.82, −0.90)	Δ_5
$9\beta^2 + 3\beta - 4$	48.18	(0.44, −0.61)	Δ_5

Table A4. (Continued.)

Quasicrystal point x	Approximate value of x	Approximate value of (x', x'')	Distance to the right neighbour
$11\beta^2+3\beta-5$	57.28	(0.06, −0.32)	Δ_4
$12\beta^2+4\beta-5$	64.57	(0.92, −0.48)	Δ_2
$13\beta^2+4\beta-6$	68.61	(0.23, −0.84)	Δ_3
$14\beta^2+4\beta-6$	73.66	(0.54, −0.20)	Δ_6
$16\beta^2+5\beta-7$	85.02	(0.703, −0.719)	Δ_5

Table A5. Candidates for tiles for quasicrystals $\Sigma([0,l_1),[0,l_2))$ with $(l_1,l_2)\in\mathcal{L}_{\mathrm{RA}}$.

Number	Tile	Signature	Number	Tile	Signature
1	β	$+-$	25	$9\beta^2+\beta-4$	$-+$
2	β^2-1	$--$	26	$9\beta^2+2\beta-4$	$-+$
3	β^2	$++$	27	$9\beta^2+3\beta-4$	$+-$
4	$\beta^2+\beta$	$+-$	28	$10\beta^2+2\beta-5$	$--$
5	$2\beta^2-1$	$-+$	29	$10\beta^2+2\beta-4$	$++$
6	$2\beta^2+\beta-1$	$+-$	30	$10\beta^2+3\beta-5$	$--$
7	$3\beta^2-1$	$-+$	31	$10\beta^2+3\beta-4$	$++$
8	$3\beta^2+\beta-2$	$--$	32	$11\beta^2+2\beta-5$	$-+$
9	$3\beta^2+\beta-1$	$++$	33	$11\beta^2+3\beta-5$	$+-$
10	$4\beta^2-2$	$-+$	34	$12\beta^2+3\beta-6$	$--$
11	$4\beta^2+\beta-2$	$--$	35	$12\beta^2+3\beta-5$	$++$
12	$4\beta^2+\beta-1$	$++$	36	$13\beta^2+2\beta-6$	$-+$
13	$5\beta^2+\beta-3$	$--$	37	$12\beta^2+4\beta-5$	$+-$
14	$5\beta^2+\beta-2$	$++$	38	$13\beta^2+3\beta-6$	$--$
15	$5\beta^2+2\beta-2$	$+-$	39	$13\beta^2+3\beta-5$	$++$
16	$6\beta^2+\beta-3$	$-+$	40	$13\beta^2+4\beta-6$	$+-$
17	$6\beta^2+2\beta-3$	$--$	41	$14\beta^2+3\beta-6$	$-+$
18	$6\beta^2+2\beta-2$	$++$	42	$14\beta^2+4\beta-6$	$+-$
19	$7\beta^2+\beta-3$	$-+$	43	$15\beta^2+3\beta-7$	$-+$
20	$7\beta^2+2\beta-3$	$+-$	44	$15\beta^2+4\beta-7$	$--$
21	$8\beta^2+\beta-4$	$-+$	45	$15\beta^2+4\beta-6$	$++$
22	$7\beta^2+3\beta-3$	$+-$	46	$16\beta^2+3\beta-7$	$-+$
23	$8\beta^2+2\beta-4$	$--$	47	$16\beta^2+4\beta-8$	$--$
24	$8\beta^2+2\beta-3$	$++$	48	$16\beta^2+4\beta-7$	$++$

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